In microfluidics, usually one manufactures rectangular pipes where the hydraulic resistance follows a more complicated expression that depends on the aspect ratio. This document lists the hydraulic resistance for channels with rectangular cross section.

2.2 Fully developed laminar flow in conduits with rectangular cross-section

Shah & London[1] correlated the pressure drop of fully developed laminar flow in rectangular conduits. Data is available only for a few aspect ratios. I extended the data base to include additional aspect ratios. The cartesian coordinates are depicted in Fig. 2.1. The momentum equation for fully developed, incompressible, time-independent flow is:

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{1}{\mu}\frac{dP}{dx} \tag{2.1}$$

where $0<y<w$ and $0<z<h$. The boundary conditions at the solid walls are $u(0, z)=u(w, z)=u(y, 0)=u(y, h)=0$.

Equation (2.1) can be readily solved by separation of variables,

$$\begin{aligned}u(y,z)=&\frac{h}{2(w+h)}\frac{1}{\mu}\frac{dP}{dx}y(y-w)+\frac{w}{2(w+h)}\frac{1}{\mu}\frac{dP}{dx}z(z-h)+\\&\sum_{n=1,3,5...}^{\infty}a_n\left(\sinh\left(\frac{n\pi}{w}z\right)+\sinh\left(\frac{n\pi}{w}(h-z)\right)\right)\sin\left(\frac{n\pi}{w}y\right)+\\&\sum_{n=1,3,5...}^{\infty}b_n\left(\sinh\left(\frac{n\pi}{h}y\right)+\sinh\left(\frac{n\pi}{h}(w-y)\right)\right)\sin\left(\frac{n\pi}{h}z\right)\end{aligned} \tag{2.2}$$

where, $a_n\sinh(\frac{n\pi}{w}h)=\frac{4hw^2}{n^3\pi^3(w+h)}\frac{1}{\mu}\frac{dP}{dx}$ and $b_n\sinh(\frac{n\pi}{h}w)=\frac{4wh^2}{n^3\pi^3(w+h)}\frac{1}{\mu}\frac{dP}{dx}$.

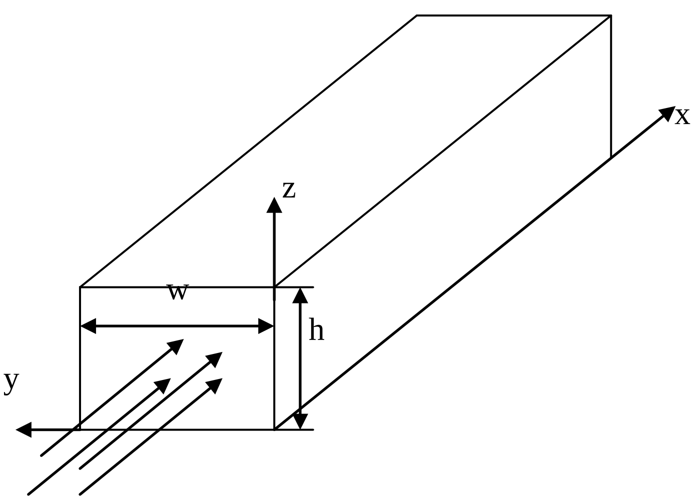


Fig. 2.1: The coordinate system for a rectangular duct.

Table 2.1: The Poiseuille number ($Po=f_xRe$) as a function of the aspect ratio α for fully developed flow in a conduit with rectangular cross-section.

α	Po	α	Po	α	Po	α	Po	α	Po
0.005	23.6517	0.205	18.9804	0.405	16.3199	0.605	14.9570	0.805	14.3695
0.01	23.5874	0.21	18.8923	0.41	16.2727	0.61	14.9345	0.81	14.3616
0.015	23.4623	0.215	18.8054	0.415	16.2262	0.615	14.9126	0.815	14.3539
0.02	23.3235	0.22	18.7199	0422	16.1805	0.622	14.8911	0.822	14.3465
0.025	23.1803	0.225	18.6356	0425	16.1356	0.625	14.8705	0.825	14.3393
0.03	23.0360	0.23	18.5525	0.43	16.0915	0.63	14.8495	0.83	14.3324
0.035	22.8921	0.235	18.4707	0.435	16.0481	0.635	14.8294	0.835	14.3258
0.04	22.7491	0.24	18.3901	0.44	16.0054	0.64	14.8097	0.84	14.3194
0.045	22.6075	0.245	18.3106	0.445	15.9635	0.645	14.7904	0.845	14.3133
0.05	22.4674	0.25	18.2324	0.45	15.9223	0.65	14.7715	0.85	14.3074
0.055	22.3291	0.255	18.1553	0.455	15.8818	0.655	14.7531	0.855	14.3017
0.06	22.1926	0.26	18.0794	0.46	15.8420	0.66	14.7351	0.86	14.2963
0.065	22.0578	0.265	18.0046	0.465	15.8029	0.665	14.7175	0.865	14.2911
0.07	21.9250	0.27	17.9310	0.47	15.7645	0.67	14.7003	0.87	14.2861
0.075	21.7939	0.275	17.8584	0.475	15.7267	0.675	14.6835	0.875	14.2814
0.08	21.6648	0.28	17.7870	0.48	15.6897	0.68	14.6671	0.88	14.2768
0.085	21.5375	0.285	17.7166	0.485	15.6533	0.685	14.6511	0.885	14.2725
0.09	21.4120	0.29	17.6473	0.49	15.6175	0.69	14.6354	0.89	14.2684
0.095	21.2883	0.295	17.5790	0.495	15.5824	0.695	14.6202	0.895	14.2646
0.1	21.1664	0.3	17.5118	0.5	15.5479	0.7	14.6053	0.9	14.2609
0.105	21.0463	0.305	17.4456	0.505	15.5141	0.705	14.5908	0.905	14.2574
0.11	20.9279	0.31	17.3805	0.51	15.4808	0.71	14.5766	0.91	14.2542
0.115	20.8113	0.315	17.3163	0.515	15.4482	0.715	14.5628	0.915	14.2511
0.12	20.6964	0.32	17.2531	0.52	15.4162	0.72	14.5493	0.92	14.2482
0.125	20.5831	0.325	17.1909	0.525	15.3848	0.725	14.5362	0.925	14.2456
0.13	20.4715	0.33	17.1297	0.53	15.3540	0.73	14.5234	0.93	14.2431
0.135	20.3615	0.335	17.0695	0.535	15.3237	0.735	14.5110	0.935	14.2408
0.14	20.2531	0.34	17.0101	0.54	15.2940	0.74	14.4989	0.94	14.2387
0.145	20.1463	0.345	16.9518	0.545	15.2649	0.745	14.4871	0.945	14.2368
0.15	20.0411	0.35	16.8943	0.55	15.2363	0.75	14.4756	0.95	14.2350
0.155	19.9374	0.355	16.8377	0.555	15.2083	0.755	14.4645	0.955	14.2335
0.16	19.8353	0.36	16.7821	0.56	15.1809	0.76	14.4536	0.96	14.2321
0.165	19.7346	0.365	16.7273	0.565	15.1539	0.765	14.4431	0.965	14.2309
0.17	19.6354	0.37	16.6734	0.57	15.1275	0.77	14.4329	0.97	14.2298
0.175	19.5376	0.375	16.6204	0.575	15.1017	0.775	14.4230	0.975	14.2290
0.18	19.4413	0.38	16.5682	0.58	15.0763	0.78	14.4133	0.98	14.2282
0.185	19.3464	0.385	16.5169	0.585	15.0514	0.785	14.4040	0.985	14.2277
0.19	19.2529	0.39	16.4664	0.59	15.0271	0.79	14.3950	0.99	14.2273
0.195	19.1607	0.395	16.4168	0.595	15.0032	0.795	14.3862	0.995	14.2271
0.2	19.0699	0.4	16.3679	0.6	14.9799	0.8	14.3777	1.0	14.2270

$$\frac{\Delta P}{\Delta L} = 2\frac{Po}{Re}\frac{\rho U^2}{D_h}$$

In planar Poiseuille flow, $\alpha=0$ and $Po=24$.
$f_D=4f_x$, f_x: the friction coefficient; f_D: Darcy friction factor.
For circular pipe, $Po=16$.

Witness that the pressure drop is proportional to the flow rate. I define the friction coefficient $f_x=\frac{\tau_w}{\frac{1}{2}\rho U^2}$, where $\tau_w=\frac{1}{4}D_h\frac{dP}{dx}$ is the average shear at the conduit's wall; U is the cross-sectional average fluid velocity; and D_h is the conduit's hydraulic diameter. The Poiseuille number, $Po=f_x Re=\frac{2D_h\tau_w}{\mu U}$, depends only on the conduit's geometry and is independent of the Reynolds number. Table 1 documents Po as a function of the cross-section's aspect ratio $\alpha=\frac{w}{h}$.

2.3 Entrance effects

The equations governing the time-independent, incompressible developing flow in the conduit's entrance region ($x<L_{hy}\sim 0.1D_h Re$) are,

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+\frac{dP}{dx}=\mu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{2.3}$$

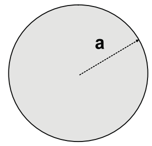
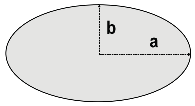
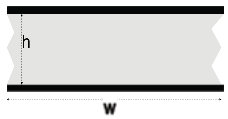
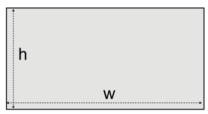
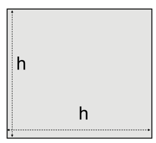
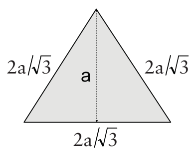
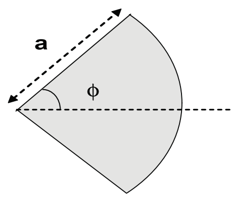
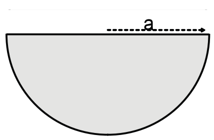
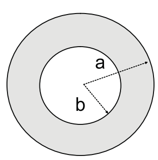
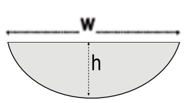
and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.4}$$

The hydraulic entrance effects cause extra pressure losses. An apparent friction coefficient is defined as $f_{app}=\frac{D_h}{x}\frac{\Delta P_x}{\frac{1}{2}\rho U^2}=4f_x+\frac{K(x)}{Re}$. K(x) is referred to as the "pressure defect" due to the hydrodynamic entrance effect and it decreases monotonically as x increases. In the fully developed region, $K(x)\rightarrow 0$. Shah & London[1] correlated $f_{app}Re$ as a function of both Reynolds number and x.

1. Shah, R., and London, A., L., 1978, Laminar Flow Forced Convection in Ducts, Academic Press, New York.
2. Yi, Mingqiang, "Fluid flow and mixing in micron -size conduits" (2001). Dissertations available from ProQuest. AAI3003711. https://repository.upenn.edu/dissertations/AAI3003711

Various Cross Section Shapes

Shape	*Geometry*	*Expression* $\Delta P = RV$	*Area, Perimeter*	*Reference*
Circle		$R = \frac{8\mu L}{a^2}$	$S = \pi a^2$ $p = 2\pi R$	[1]
Ellipse		$R = \frac{4\mu L\left(1+\varepsilon^2\right)}{b^2}$	$S = \pi a b$ $p = 4 a E\sqrt{1-\left(b/a\right)^2}$	[2]
Parallel plates		$R = \frac{12\mu L}{h^2}$	$S = h w$ $p = 2w$	[1]
Rectangle		$R = \frac{4\mu L}{h^2}\frac{1}{q(\varepsilon)}$ $q(\varepsilon) = \frac{1}{3} - \frac{64}{\pi^5}\varepsilon\tanh\left(\frac{\pi}{2\varepsilon}\right)$	$S = h w$ $p = 2\left(w+h\right)$	[2]
Square		$R = \frac{28.3\mu L}{h^2}$	$S = h^2$ $p = 4h$	[2, 3]
Triangle (equilateral)		$R = \frac{60\mu L}{a^2}$	$S = a^2/\sqrt{3}$ $p = 6a/\sqrt{3}$	[2, 3]
Circular sector		$R = \frac{\mu L}{a^2}\frac{1}{g(\phi)}$	$S = \phi a^2$ $p = 2a(1+\phi)$	[2]
Half-circle		$R = \frac{64\,\mu L}{3a^2}$	$S = \pi a^2/2$ $p = \pi a$	
Annular torus		$R = \frac{8\mu L}{a^2}\frac{1}{\beta}$ $\beta = \varepsilon^2 - 1 + \frac{2\ln(1/\varepsilon)+\varepsilon^2-1}{\ln(1/\varepsilon)}$	$S = \pi\left(a^2-b^2\right)$ $p = 2\pi\left(a+b\right)$	[2]
Parabola		$R = \frac{35\mu L}{4h^2}$	$S = \frac{hw}{3}$	[3]

$\varepsilon = b/a$ or $\varepsilon = h/w$ and $g(\phi) = \frac{\tan(2\phi)\,2-\phi}{16\phi} - \frac{128\phi^3}{\pi^5}\sum_1^\infty\left[\frac{1}{(2n-1)^2(2n-1+4\phi/\pi)^2(2n-1-4\phi/\pi)}\right]$

E is the complete elliptic integral of the second kind. When two dimensions are present, ε is their ratio ($\varepsilon < 1$).

1. Shah, R. K., and A.L. London, "Laminar flow forced convection in ducts," Academic Press, 1978, pp 197.
2. Bahrami, M., M. M. Yovanovich, and J. R. Culham, "Pressure Drop of Fully-Developed, Laminar Flow in Microchannels of Arbitrary Cross Section," *Proceedings of ICMM 2005, 3rd International Conference on Microchannels and Minichannels*, Toronto, Ontario, Canada, June 13–15, 2005.
3. Bruus, H., *Theoretical Microfluidics*, Oxford, U.K.: Oxford Master Series in Condensed Matter Physics, 2008.